Consequences of Special Relativity

- The Twins Paradox
- Length Contraction
- Relativistic Momentum
- Relativistic Energy
- Homework
The Twins Paradox 1

- Intriguing biological consequence of time dilation
- Consider a set of 20 year old twins named Speedo and Goslo.
- Speedo takes a round trip to a planet 20 light years from Earth on a spaceship with a speed of 0.95c.
- When he returns to Earth, he finds that his brother has aged 42 years while he has aged only 13 years.
- But from Speedo’s perspective, he was at rest while Earth took a 13-year round trip at a speed of 0.95c
- This leads to an apparent contradiction - each twin observed the other in motion and might claim that the other’s clock runs slow.
- Which twin has actually aged more?
The Twins Paradox 2

- Actually, Speedo does return younger than Goslo.
- The situation is not really symmetric.
- Speedo must experience accelerations in leaving Earth, turning around, and arriving back at Earth, and therefore does not remain in the same inertial reference frame.
- So, only Goslo, who is in a single inertial frame, can apply the time dilation equation to Speedo’s trip.
- Thus, Goslo finds that instead of aging 42 years, Speedo ages only

\[
\Delta t_p = \Delta t \sqrt{1 - \left(\frac{v}{c}\right)^2}
\]

\[
\Delta t_p = (42 \text{ yrs}) \sqrt{1 - \left(\frac{0.95c}{c}\right)^2} = 13 \text{ yrs}
\]
Length Contraction

- Consider two observers - Sally, seated on a train moving through a station, and Sam, on the station platform.

- Sam, using a tape measure, finds the length of the platform to be \( L_p \), a proper length because the platform is at rest with respect to him.

- Sam also notes that Sally, on the train, moves through the length in a time \( \Delta t = \frac{L_p}{v} \), where \( v \) is the speed of the train, so

\[
L_p = v \Delta t \quad (Sam)
\]

- Sally would measure the length of the platform to be

\[
L = v \Delta t_p \quad (Sally)
\]

\[
\frac{L}{L_p} = \frac{v \Delta t_p}{v \Delta t} = \frac{1}{\gamma}
\]

\[
L = \frac{L_p}{\gamma}
\]
Example 1

A meter stick in frame $S'$ makes an angle of $30^\circ$ with the $x'$ axis. If that frame moves parallel to the $x$ axis of frame $S$ with a speed $0.90c$ relative to frame $S$, what is the length of the stick as measured from $S$?
Example 1 Solution

A meter stick in frame $S'$ makes an angle of $30^\circ$ with the $x'$ axis. If that frame moves parallel to the $x$ axis of frame $S$ with a speed $0.90c$ relative to frame $S$, what is the length of the stick as measured from $S$?

\[ l_{x'} = (1 \text{ m}) \cos 30^\circ = 0.866 \text{ m} \]
\[ l_{y'} = (1 \text{ m}) \sin 30^\circ = 0.5 \text{ m} \]

\[
l_x = l_{x'} \gamma = l_{x'} \sqrt{1 - \left(\frac{v}{c}\right)^2} = 0.866 \text{ m} \sqrt{1 - \left(\frac{0.90c}{c}\right)^2} = 0.377 \text{ m}
\]
\[
l_y = l_{y'} = 0.5 \text{ m}
\]
\[
l = \sqrt{l_x^2 + l_y^2} = \sqrt{(0.377 \text{ m})^2 + (0.5 \text{ m})^2} = 0.63 \text{ m}
\]
Relativistic Momentum

- Suppose that two observers, each in a different inertial reference frame, watch an isolated collision between two particles.

- We know that - even though the two observers measure different velocities for the colliding particles - they find that the law of conservation of momentum holds.

- However, in relativistic collisions we find that if we continue to define the momentum as the product of the mass and the velocity, that momentum is not conserved for all inertial observers.

- In order to conserve momentum in all inertial reference frames, we must redefine the momentum of a particle as the product of the Lorentz factor, $\gamma$, the rest mass, $m$, and the velocity, $u$, of the particle.

\[ p = \gamma m u \]
Example 2

An electron, which has a mass of $9.11 \times 10^{-31}$ kg, moves with a speed of $0.75c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.
Example 2 Solution

An electron, which has a mass of $9.11 \times 10^{-31}$ kg, moves with a speed of $0.75c$. Find its relativistic momentum and compare this with the momentum calculated from the classical expression.

\[
p_{rel} = \frac{m_e u}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{(9.11 \times 10^{-31} \text{kg}) (0.75) (3.00 \times 10^8 \text{m/s})}{\sqrt{1 - \frac{(0.75)^2 c^2}{c^2}}} = 3.10 \times 10^{-22} \text{kg} \cdot \text{m/s}
\]

\[
p_{cl} = m_e u = (9.11 \times 10^{-31} \text{kg}) (0.75) (3.00 \times 10^8 \text{m/s}) = 2.05 \times 10^{-22} \text{kg} \cdot \text{m/s}
\]

\[
\frac{p_{rel} - p_{cl}}{p_{rel}} = 0.34
\]

The classical expression gives a momentum 34% lower than the relativistic expression.
Relativistic Energy

- Einstein showed the equivalence of mass and energy.
- The energy associated with the mass of a particle is called the rest energy, $E_R$.
  \[ E_R = mc^2 \]
- The total energy of a particle, $E$, is the sum of its rest energy and its kinetic energy.
  \[ E = E_R + K = mc^2 + K \]
- The total energy can also be written as
  \[ E = \gamma mc^2 \]
Relativistic Kinetic Energy

- Relativistic kinetic energy can be expressed as
  \[ K = E - mc^2 = \gamma mc^2 - mc^2 = mc^2 (\gamma - 1) \]

- The total energy can also be written as
  \[ E^2 = (pc)^2 + (mc^2)^2 \]

- For a massless particle
  \[ E = pc \]
Example 3

(a) What is the total energy $E$ of a 2.53 MeV electron?
(b) What is the magnitude $p$ of the electron’s momentum, in the unit MeV/c?
Example 3 Solution

(a) What is the total energy $E$ of a 2.53 MeV electron?

$$E = mc^2 + K$$

$$mc^2 = (9.109 \times 10^{-31} \text{kg}) (2.998 \times 10^8 \text{m/s})^2 = 8.187 \times 10^{-14} \text{J}$$

$$mc^2 = 8.187 \times 10^{-14} \text{J} \left(\frac{1 \text{ MeV}}{1.602 \times 10^{-13} \text{J}}\right) = 0.511 \text{ MeV}$$

$$E = 0.511 \text{ MeV} + 2.53 \text{ MeV} = 3.04 \text{ MeV}$$

(b) What is the magnitude $p$ of the electron’s momentum, in the unit MeV/c?

$$pc = \sqrt{E^2 - (mc^2)^2} = \sqrt{(3.04 \text{ MeV})^2 - (0.511 \text{ MeV})^2} = 3.00 \text{ MeV}$$

$$p = 3.00 \text{ MeV/c}$$
Homework Set 17 - Due Fri. Oct. 22

- Read Sections 9.6 - 9.8
- Answer question 9.5 & 9.12